# Problem 3.31

**Legendre polynomials.** Use the Gram–Schmidt procedure (Problem A.4) to orthonormalize the functions 1, $x$, $x^2$, and $x^3$, on the interval $-1 \leq x \leq 1$. You may recognize the results—they are (apart from normalization)[39] **Legendre polynomials** (Problem 2.64 and Table 4.1).

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### Solution

Let the first function be

$$f_1(x) = 1.$$

Divide it by its norm to get the first orthonormal function.

$$\boxed{e_1(x) = \frac{f_1(x)}{\|f_1(x)\|} = \frac{1}{\sqrt{\int_{-1}^{1} 1^2\,dx}} = \frac{1}{\sqrt{2}}}$$

Let the second function be

$$f_2(x) = x.$$

Subtract the component of $f_2(x)$ parallel to $e_1(x)$ off of $f_2(x)$ to get a function orthogonal to $e_1(x)$.

$$\begin{aligned} f_2(x) - \langle e_1 \,|\, f_2\rangle e_1(x) &= x - \left(\frac{1}{\sqrt{2}}\right)\int_{-1}^{1}\left(\frac{1}{\sqrt{2}}\right) x\,dx \\ &= x - \frac{1}{2}\underbrace{\int_{-1}^{1} x\,dx}_{=\,0} \\ &= x \end{aligned}$$

Divide it by its norm to get the second orthonormal function.

$$\boxed{e_2(x) = \frac{x}{\|x\|} = \frac{x}{\sqrt{\int_{-1}^{1} x^2\,dx}} = \frac{x}{\sqrt{\frac{2}{3}}} = \sqrt{\frac{3}{2}}x}$$

Let the third function be

$$f_3(x) = x^2.$$

Subtract the component of $f_3(x)$ parallel to $e_1(x)$ and the component of $f_3(x)$ parallel to $e_2(x)$ off of $f_3(x)$ to get a function orthogonal to $e_1(x)$ and $e_2(x)$.

$$\begin{aligned} f_3(x) - \langle e_1 \,|\, f_3\rangle e_1(x) - \langle e_2 \,|\, f_3\rangle e_2(x) &= x^2 - \left(\frac{1}{\sqrt{2}}\right)\int_{-1}^{1}\left(\frac{1}{\sqrt{2}}\right) x^2\,dx - \left(\sqrt{\frac{3}{2}}x\right)\int_{-1}^{1}\left(\sqrt{\frac{3}{2}}x\right) x^2\,dx \\ &= x^2 - \frac{1}{2}\int_{-1}^{1} x^2\,dx - \frac{3}{2}x\underbrace{\int_{-1}^{1} x^3\,dx}_{=\,0} \end{aligned}$$

[39] Legendre didn't know what the best convention would be; he picked the overall factor so that all his functions would go to 1 at $x = 1$, and we're stuck with his unfortunate choice.

Evaluate the integral.

$$\begin{aligned} f_3(x) - \langle e_1 \mid f_3 \rangle e_1(x) - \langle e_2 \mid f_3 \rangle e_2(x) &= x^2 - \frac{1}{2}\left(\frac{2}{3}\right) \\ &= x^2 - \frac{1}{3} \end{aligned}$$

Divide it by its norm to get the third orthonormal function.

$$\boxed{e_3(x) = \frac{x^2 - \frac{1}{3}}{\left\| x^2 - \frac{1}{3} \right\|} = \frac{x^2 - \frac{1}{3}}{\sqrt{\int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx}} = \frac{x^2 - \frac{1}{3}}{\sqrt{\frac{8}{45}}} = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)}$$

Let the fourth function be

$$f_4(x) = x^3.$$

Subtract the component of $f_4(x)$ parallel to $e_1(x)$ and the component of $f_4(x)$ parallel to $e_2(x)$ and the component of $f_4(x)$ parallel to $e_3(x)$ off of $f_4(x)$ to get a function orthogonal to $e_1(x)$ and $e_2(x)$ and $e_3(x)$.

$$\begin{aligned} f_4(x) - \langle e_1 \mid f_4 \rangle e_1(x) - \langle e_2 \mid f_4 \rangle e_2(x) - \langle e_3 \mid f_4 \rangle e_3(x) &= x^3 - \left(\frac{1}{\sqrt{2}}\right) \int_{-1}^{1} \left(\frac{1}{\sqrt{2}}\right) x^3 \, dx - \left(\sqrt{\frac{3}{2}}x\right) \int_{-1}^{1} \left(\sqrt{\frac{3}{2}}x\right) x^3 \, dx \\ &\quad - \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) \int_{-1}^{1} \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) x^3 \, dx \\ &= x^3 - \frac{1}{2} \underbrace{\int_{-1}^{1} x^3 \, dx}_{=0} - \frac{3}{2}x \int_{-1}^{1} x^4 \, dx - \frac{45}{8}\left(x^2 - \frac{1}{3}\right) \underbrace{\int_{-1}^{1} \left(x^2 - \frac{1}{3}\right) x^3 \, dx}_{=0} \\ &= x^3 - \frac{3}{2}x\left(\frac{2}{5}\right) \\ &= x^3 - \frac{3}{5}x \end{aligned}$$

Divide it by its norm to get the fourth orthonormal function.

$$\boxed{e_4(x) = \frac{x^3 - \frac{3}{5}x}{\left\| x^3 - \frac{3}{5}x \right\|} = \frac{x^3 - \frac{3}{5}x}{\sqrt{\int_{-1}^{1} \left(x^3 - \frac{3}{5}x\right)^2 dx}} = \frac{x^3 - \frac{3}{5}x}{\sqrt{\frac{8}{175}}} = \sqrt{\frac{175}{8}}\left(x^3 - \frac{3}{5}x\right)}$$